Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce

Janet Heine Barnett[*]

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1 Introduction

On virtually the same day in 1847, two major new works on logic were published by prominent British mathematicians. Augustus De Morgan (1806–1871) opened his *Formal Logic* with the following description of what is known today as 'logical validity' [6, p. 1]:

> The first notion which a reader can form of Logic is by viewing it as the examination of that part of reasoning which depends upon the manner in which inferences are formed, and the investigation of general maxims and rules for constructing arguments, so that the conclusion may contain no inaccuracy which was not previously asserted in the premises. It has so far nothing to do with the truth of the facts, opinions, or presumptions, from which an inference is derived; but simply takes care that the inference shall certainly be true, if the premises be true. . . . Whether the premises be true or false, is not a question of logic, but of morals, philosophy, history, or any other knowledge to which their subject-matter belongs: the question of logic is, does the conclusion certainly follow if the premises be true?

The second of these new nineteenth century works on logic was *The Mathematical Analysis of Logic* by George Boole (1815–1864). Like De Morgan, Boole sought to stretch the boundaries of traditional syllogistic[1] logic by developing a general method for representing and manipulating *all* logically valid inferences or, as DeMorgan described it to Boole in a letter dated 28 November 1847, to develop 'mechanical modes of making transitions, with a notation which represents our head work' [22, p. 25]. In contrast to De Morgan, however, Boole took the significant step of explicitly adopting *algebraic* methods for this purpose. As De Morgan himself later proclaimed, "Mr. Boole's generalization of the forms of logic is by far the boldest and most original . . . " (as quoted in [15, p. 174]).

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[*]Department of Mathematics and Physics; Colorado State University-Pueblo; Pueblo, CO 81001 - 4901; janet.barnett@colostate-pueblo.edu.

[1]A syllogism is a particular type of logical form consisting of three statements – a major premise, a minor premise (or middle term), and a conclusion based on the two premises – which was first studied by the ancient Greek philosopher Aristotle (384–322 B.C.E.). The following example of a particular syllogistic form is found in the opening paragraph of De Morgan's *Formal Logic*:

All men will die.	Every Y is X.
All men are rational beings.	Every Y is Z.
Therefore some rational beings will die.	Therefore some Zs are Xs.

1

This project examines Boole's bold and original approach to logic and the strange new system of algebra which developed out of it. Through his own writings, we will consider the ways in which the laws of this new algebraic system both resemble and differ from those of standard algebra, as well as the reasons why it satisfies these various laws. The project then follows refinements made to Boole's 'Algebra of Logic' by two of the many later mathematicians who studied it: John Venn (1834–1923) and C. S. Peirce (1839–1914). Through their writing, we will gain insight into the concepts of the logic of classes (or what might be referred to today as elementary set theory) and into the rationale behind the choices these mathematicians made in developing these concepts. Along the way, project questions interspersed between these readings explore a variety of mathematical themes, such as the notion of an inverse operation, the concept of duality, issues related to mathematical notation, and standards of rigor and proof. In the closing section of the project, we summarize both how Boole's original 'Algebra of Logic' relates to elementary set theory as it is is typically presented today, and how that elementary set theory, when viewed as an algebraic system, relates to the abstract axiomatic structure known today as a *boolean algebra* which developed in the early 20th century as Boole's original 'Algebra of Logic' gained its independence from the logic of classes[2].

2 Boole's *Laws of Thought*: The Basic Elements

Born in Lincoln England in 1815, Boole was highly influenced by the love of learning which his father, a shoemaker of humble means, passed on to him. His early scholarly interest was in languages – he learned Latin, Greek, French, Italian and German largely through self study. At the age of 17, however, he discovered mathematics and began to study works of great mathematicians on his own. Boole's lack of formal academic credentials prevented him from obtaining a university post for many years. During this period, he conducted mathematical research while working as a school teacher. In 1842, he began a lifelong correspondance with fellow mathematician and logician Augustus De Morgan; their approaches to logic were, however, developed quite independently [22]. In fact, Boole's earliest mathematical publications lay outside the field of mathematical logic. In 1844, he was awarded a medal by the Royal Society for an article on new techniques in differential equations. He also wrote two well-received textbooks on differential equations, after finally obtaining a university post in Cork, Ireland in 1849. It was at Cork that Boole conducted the investigations of logic which led to his mature work on logic, *An Investigation of the Laws of Thought*, published in 1854. Ironically, the book was not initially well received; Boole and a friend who bore the expense of its initial printing probably did not recover their costs. Although he wrote several unpublished manuscripts on logic following the appearance of *Laws of Thought*, and planned another book as its sequel, all of his later publications were on differential equations. At the age of 49, he died of pleuro-pneumonia contracted after walking home 3 miles in a rain storm. His wife Mary Everest Boole (1832–1916), herself a mathematician and teacher, and their five daughters, two of whom later became mathematicians, survived him.

Boole's work in logic was strongly influenced by the general state of British mathematics at the time. Over the course of the eighteenth century, British mathematics had become cut off from that of the European continent, in part due to a priority controversy concerning the invention of calculus by Gottfried Leibniz (1646–1716) and Isaac Newton (1643–1727). On the continent, the differential techniques and notation of Leibniz allowed mathematicians to make significant advances

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[2]For further details on early twentieth century work related to the axiomatization of boolean algebra, see the project "Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization," Janet Barnett author. Readers interested in a contemporary application of boolean algebra are also referred to the project "Applying Boolean Algebra to Circuit Design: Claude Shannon," Janet Barnett author.

in the development of calculus. In comparison, British mathematics appeared almost stagnant. By the start of the nineteenth century, however, significant changes began to emerge in English society as a result of the Industrial Revolution. In particular, the role of universities in promoting research and science became a subject of concern. One outcome of these changes for mathematics was the formation of 'The Analytic Society' by a group of mathematicians at Cambridge in 1812. As members of the Society began to import continental methods of symbolic manipulation into the university curriculum, concerns about the foundations of algebra emerged, including questions about the meaning of negative and imaginary numbers. The concept of a 'symbolical algebra' was developed in response to these questions.

In contrast to 'arithmetical algebra,' which derives its laws from the actual meaning of operations on numbers, a 'symbolical algebra' begins with formal unrestricted laws for a given set of symbols and operations, and only later interprets these as having particular meaning. In his *Trigonometry and Double Algebra* [7, p. 94], De Morgan gave the following example of a symbolic algebra:

> Given symbols M, N, $+$, and one sole relation of combination, namely that $M + N$ is the same as $N + M$.

He then provided five distinct interpretations which would give meaning to this particular (commutative) algebra, ranging from

> M and N may be magnitudes, and $+$ the sign of addition of the second to the first

to

> M and N may be numbers, and $+$ the sign of multiplying the first to the second

to

> M and N may be nations, and $+$ the sign of the consequent having fought a battle with the antecedent.

In what follows, we will see the influence of these ideas on Boole's conception of logic as presented in *Laws of Thought*. We begin with excerpts from Chapter 2 [4, p. 24–38] in which Boole (a) explained the use of 'signs' to represent 'classes,' (b) defined a system of symbols $(+, -, \times, 0, 1)$ to represent operations on these signs, and (c) deduced the basic laws which these operations must follow. As you read these excerpts, respond to the project questions which are interspersed.

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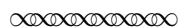
An Investigation of the Laws of Thought[3]

Chapter II

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> 1. That Language is an instrument of human reason, and not merely a medium for the expression of thought, is a truth generally admitted. ...

> 2. The elements of which all language consists are signs or symbol. Words are signs. ... But, words ... are not the only signs which we are capable of employing. Arbitrary marks, which

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[3]To set them apart from the project narrative, all original source excerpts are set in sans serif font and bracketed by the following symbol at their beginning and end: ∞∞∞∞∞∞∞∞∞∞ Both original source excerpts and tasks for the reader are indented relative to the project narrative.

speak only to the eye, and arbitrary sounds or actions, which address themselves to some other sense, are equally of the nature of signs, provided that their representative office is defined and understood. In the mathematical sciences, letters, and the symbols $+, -, =,$ &c., are used as signs .... In the present treatise ...it is with written signs that we have to do, and it is with reference to these exclusively that the term "sign" will be employed. The essential properties of signs are enumerated in the following definition:

> *Definition.* − A sign is an arbitrary mark, having a fixed interpretation, and susceptible of combination with other signs in subjection to fixed laws dependent upon their mutual interpretation.

$$\dots$$

4. The analysis and classification of those signs by which the operations of reasoning are conducted will be considered in the following Proposition:

Proposition I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz:

*1st. Literal symbols, as x, y &c., representing things as subjects of our conceptions.*

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

$$\dots$$

6. Now, as it has been defined that a sign is an arbitrary mark, it is permissible to replace all signs of the species above described by letters. Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as x. ...By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms "nothing" and "universe," which as "classes" should be understood to comprise respectively "no beings," and "all beings." ...Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for "white things," and y for "sheep," let xy stand for "white sheep;" and in like manner, if z stand for "horned things," and x and y retain their previous interpretations, let zxy represent "horned white sheep," i. e. that collection of things to which the name "sheep," and the descriptions "white" and "horned" are together applicable. Let us now consider the laws to which the symbols x , y , &c., used in the above sense, are subject.

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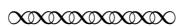
Task 1

The first "Law of Thought" described by Boole is the following:

> First, it is evident, that according to the above combinations, the order in which two symbols are written is indifferent. ... Hence we have,

$$xy = yx. \tag{1}$$

Provide an example of classes x and y, different than the example given by Boole in his section 6 above, to illustrate why this law *(i.e., commutativity)* holds for literal symbols representing classes.

Task 2

Another standard algebraic law for multiplication is associativity:

$$(xy)z = x(yz) \text{ for all } x, y, z$$

Provide an example of classes x, y and z to illustrate why this law also holds for literal symbols representing classes. Then explain how we know that Boole assumed associativity for multiplication in his section 6 above, even though he did not explicitly mention it.

Boole commented on the analogy between the law of commutativity for symbolic logic and the corresponding law for algebra in his section 8, which we shall read below. Note his insistence on viewing commutativity as a *formal* (e.g., symbolic) law which happens to hold in two distinct systems (e.g., algebra and logic) in that section. He continued in section 9 to then draw on this analogy with algebra in order to deduce another formal law of logic.

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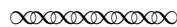
8. ... The law expressed by (1) may be characterized by saying that the literal symbols x, y, z, are *commutative, like the symbols of Algebra*. In saying this, it is not affirmed that the process of multiplication in Algebra, of which the fundamental law is expressed by the equation

$$xy = yx,$$

possesses in itself any analogy with that process of logical combination which xy has been made to represent above; but only that if the arithmetical and the logical process are expressed in the same manner, their symbolical expressions will be subject to the same formal law. The evidence of that subjection is in the two cases is quite distinct.

9. As the combination of two literal symbols in the form xy expresses the whole of that class of objects to which the names or qualities represented by x and y are together applicable, it follows that if the two symbols have exactly the same signification, their combination expresses no more than either of the symbols taken alone would do. In such case we should therefore have

$$xy = x.$$

As y is, however, supposed to have the same meaning as x, we may replace it in the above equation by x, and we thus get

$$xx = x.$$

Now in common Algebra the combination xx is more briefly represented by x^2. Let us adopt the same principle of notation here; for the mode of expressing a particular succession of mental operations is a thing in itself quite as arbitrary as the mode of expressing a single idea or operation (II. 3). In accordance with this notation, then, the above equation assumes the form

$$x^2 = x, \tag{2}$$

and is, in fact, the expression of a second general law of those symbols by which names, qualities, or descriptions, are symbolically represented.
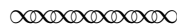
Task 3

Use a specific example of a class x to explain why the symbolic law "$x^2 = x$" makes sense as a property of classes under Boole's definition of multiplication. (Remember that xy represents the class of things which simultaneously possess the quality x and the quality y.) Does the symbolic law "$x^2 = x$" also hold in arithmetical algebra? Explain.

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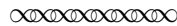
10. We pass now to the consideration of another class of the signs of speech, and of the laws connected with their use.

11. *Signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts.*

We are not only capable of entertaining the conceptions of objects, as characterized by names, qualities, or circumstances, applicable to each individual of the group under consideration, but also of forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described. For this purpose we use the conjunctions "and," "or," &c. "Trees and minerals," "barren mountains, or fertile vales," are examples of this kind. In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign $+$ in algebra, and their laws identical. Thus the expression "men and women" is, conventional meanings set aside, equivalent with the expression "women and men." Let x represent "men," y, "women;" and let $+$ stand for "and" and "or," then we have

$$x + y = y + x, \tag{3}$$

an equation which would equally hold true if x and y represented numbers, and $+$ were the sign of arithmetical addition.
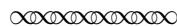
Task 4

Note that Boole imposed a restriction on the use of the addition symbol "+" in his system by asserting that "the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another." In this task, we consider the question of whether this restriction reflects standard language usage, as Boole claimed.

Consider, for instance, the following expressions:

(I) infants and teenagers   (III) lying or confused
(II) dancers and singers   (IV) conqueror of Gaul or first emperor of Rome

For which of these is the conjunction (*or* , *and*) used in the exclusive sense specified by Boole? That is, is there an implication in standard language usage that a particular individual under discussion will belong to at most one of the named classes, but not both classes simultaneously?

Note: We will refer to classes with no common members as *disjoint classes* (or *disjoint sets*). In a now-standard diagram named after John Venn about which we will say more in Section 3, two disjoint sets are represented by two non-overlapping circles. A Venn diagram for sets x, y which are not disjoint consists of two overlapping circles, with their overlapping region representing the intersection xy of the sets. The region covered by both sets together (including their overlap) represents the union $x + y$ of the sets.

Task 5

In his section 11 above, Boole referred to the analogy of symbolic logic with arithmetical algebra as further justification for restricting the use of "+" to disjoint classes:

> In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" "or" are analogous with the sign + in algebra, and their laws identical.

Recall that the standard definition of addition for whole numbers m, n defines $m + n$ to be the total number of elements in the union (or aggregate) of a set containing m objects with a set containing n objects. Provide one or more specific examples to illustrate why it is important to use disjoint sets in this definition.

Task 6

Later in his section 11, Boole justified the distributivity law,

$$z(x + y) = zx + zy, \tag{4}$$

by letting x represent "men," y represent "women," and z represent "European." Describe, in English, the meaning of each side of the equation for this particular example. Is this is a convincing justification for the law?

We now return to Boole's section 11, where he introduced another operation, denoted by the subtraction symbol "−", again using the analogy with algebra as motivation. Following a discussion of the equality symbol "=" to represent the verb *is* or *are* in his section 12 (omitted here), Boole then further explored the analogy of symbolic logic and algebra in section 13 below.

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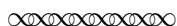
11. ...The above are the laws which govern the use of the sign $+$, here used to denote the positive operation of aggregating parts into a whole. But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done. Thus we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from the whole. This operation we express in common language by the sign *except*, as, "All men except Asiatics," "All states except those which are monarchical." Here it is implied that the things excepted form a part of the things from which they are excepted. As we have expressed the operation of aggregation by the sign $+$, so we may express the negative operation above described by $-$ minus. Thus if x be taken to represent men, and y, Asiatics, i. e. Asiatic men, then the conception of "All men except Asiatics" will be expressed by $x - y$. And if we represent by x, "states," and by y the descriptive property "having a monarchical form," then the conception of "All states except those which are monarchical" will be expressed by $x - xy$.

. . .

13. ...Let us take the Proposition, "The stars are the suns and the planets," and let us represent stars by x, suns by y, and planets by z; we have then

$$x = y + z. \tag{7}$$

Now if it be true that the stars are the suns and the planets, it will follow that the stars, except the planets, are suns. This would give the equation

$$x - z = y, \tag{8}$$

which must therefore be a deduction from (7). Thus a term z has been removed from one side of an equation to the other by changing its sign. This is in accordance with the algebraic rule of transposition.

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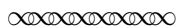
Task 7
Note Boole's comment (in his section 11) that the operation *minus* will require "that the things excepted form a part of the things from which they are excepted." For example, if x represents men and y represents women, then the expression $x - y$ is meaningless in Boole's system since the class of women does not form part of the class of men.

(a) Suppose we wanted to drop Boole's restriction in the particular example just described. What class would the expression $x - y$ denote? Based on your reading of Boole up to this point, could this class be represented symbolically in some other way within his system? Explain.

(b) Now consider the expression $d - p$ in the case where d represents drummers and p represents pianists. Explain why Boole would consider the expression $d - p$ to be meaningless, then describe the class which the expression $d - p$ would denote if Boole's restriction were removed. Based on your reading of Boole so far, how could this class be represented symbolically within his system? Explain.

(c) Based on these examples, do you agree or disagree with Boole's restriction, and why? What argument, if any, does he give for adopting this restriction in his section 11?

Task 8
Now consider Boole's equations (7) and (8) in his section 13 on the previous page, and recall his earlier restriction that "+" can only be applied to classes which share no members. Suppose we were to drop this restriction, and let y represent men, z represent doctors, and $x = y + z$ as in Boole's equation (7). Can we still deduce Boole's equation (8) in this case? That is, does $x - z = y$? Explain.

We now continue our reading of Section 13 of Boole's Chapter II.

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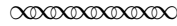
13. ... Again: If two classes of things, x and y, be identical, that is, if all members of the one are members of the other, then those members of the one class which possess a given property z will be identical with those members of the other which possess the same property z. Hence if we have the equation

$$x = y,$$

then whatever class or property z may represent, we have also

$$zx = zy.$$

This is formally the same as the algebraic law:– If both members of an equation are multiplied by the same quantity, the products are equal.

In like manner it may be shown that if the corresponding members of two equations are multiplied together, the resulting equation is true.

14. Here, however, the analogy of the present system with that of algebra, as commonly stated, appears to stop. Suppose it true that those members of a class x which possess a certain property z are identical with those members of a class y which possess the same property z, it does not follow that the members of the class x universally are identical with the members of the class y. Hence it cannot be inferred from the equation

$$zx = zy$$

that the equation

$$x = y$$

is also true. In other words, the axiom of algebraists, that both sides of an equation may be divided by the same quantity, has no formal equivalent here.

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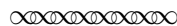
Task 9
Provide a specific example of classes x , y and z to illustrate that Boole's assertion in section 14 above is correct; that is, show that it is possible to have $zx = zy$ for $x \neq y$ and $z \neq 0$.

In the closing section (15) of Chapter II of *Laws of Thought*, Boole identified a special 'algebra of number' with properties which are strongly analogous to those exhibited by operations on classes. In Chapter III, Boole then derived several additional laws for symbolic logic from the analogy with this specific "arithmetical algebra." Our reading from Chapter III [4, p. 47–51] focuses on three specific propositions in which he provided interpretations for the symbols "0", "1", and "$1 - x$", and presented the first formal deduction of a logical principle in mathematical form.

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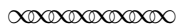
15. ...We have seen that the symbols of Logic are subject to the special law,

$$x^2 = x.$$

Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2 = 0$, and that $1^2 = 1$; and the equation $x^2 = x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity admitting only of the values 0 and 1. Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

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Task 10
Recall that for classes, the expression '$1 + 1$' is meaningless for Boole. (Why?) Suppose we dropped Boole's restriction on the use of '$+$'. Would it make sense to assign either of the two values (0 or 1) to the expression $1 + 1$ as a statement about classes? Explain. Then consider the expression '$1 + 1$' as a *numerical* statement within the 'Algebra of Number;' does it make sense to assign either 0 or 1 to the sum $1 + 1$? Why or why not?

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CHAPTER III

PROPOSITION II

To determine the logical value and significance of the symbols 0 and 1.

The symbol 0, as used in Algebra, satisfies the following formal law,

$$0 \times y = 0, \text{ or } 0y = 0, \tag{1}$$

whatever number y may represent. That this formal law may be obeyed in the system of Logic, we must assign to the symbol 0 such an interpretation that the class represented by $0y$ may be identical with the class represented by 0, whatever the class y may be. A little

consideration will show that this condition is satisfied if the symbol 0 represent Nothing.
. . .

Secondly, The symbol 1 satisfies in the system of Number the following law, viz.,

$$1 \times y = y, \text{ or } 1y = y,$$

whatever number y may represent. And this formal equation being assumed as equally valid in the system of this work, in which 1 and y represent classes, it appears that the symbol 1 must represent such a class that all the individuals which are found in any proposed class y are also all the individuals $1y$ that are common to that class y and the class represented by 1. A little consideration will here show that the class represented by 1 must be "the Universe," since this is the only class in which are found all the individuals that exist in any class. Hence the respective interpretations of the symbols 0 and 1 in the system of Logic are Nothing and Universe.

. . .

PROPOSITION III

If x represent any class of objects, then will $1 - x$ represent the contrary or supplementary class of objects, i. e. the class including all objects which are not comprehended in the class x.

For greater distinctness of conception let x represent the class men, and let us express, according to the last Proposition, the Universe by 1; now if from the conception of the Universe, as consisting of "men" and "not-men," we exclude the conception of "men," the resulting conception is that of the contrary class, "not-men." Hence the class "not-men" will be represented by $1-x$. And, in general, whatever class of objects is represented by the symbol x, the contrary class will be expressed by $1-x$. . . .

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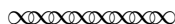
Task 11
Recall the classes used in project question 7 above: n represents infants , t represents teenagers, d represents drummers, and p represents pianists. Again suppose that we wanted to drop Boole's restriction that subtraction is only meaningful if "the things excepted form a part of the things from which they are excepted." Using the notation introduced in Propositions II and III above, how else could the classes denoted by the expression $n - t$ and the expression $d - p$ be represented symbolically? Explain briefly.

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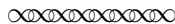
PROPOSITION IV

That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$.

Let us write this equation in the form
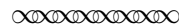
$$x - x^2 = 0,$$

whence we have

$$x(1 - x) = 0; \tag{1}$$

both these transformations being justified by the axiomatic laws of combination and transposition (II. 13). Let us, for simplicity of conception, give to the symbol x the particular interpretation of men, then $1 - x$ will represent the class of "not-men" (Prop. III.) Now the formal product of the expressions of two classes represents that class of individuals which is common to them both (II. 6). Hence $x(1 - x)$ will represent the class whose members are at once "men" and "not-men," and the equation (1) thus express the principle, *that a class whose members are at the same time men and not men does not exist*. In other words, that *it is impossible for the same individual to be at the same time a man and not a man*. Now let the meaning of the symbol x be extended from the representing of "men," to that of any class of beings characterized by the possession of any quality whatever; and the equation (1) will then express that it is impossible for a being to possess a quality and not to possess that quality at the same time. But this is identically that "principle of contradiction" which Aristotle has described as the fundamental axiom of all philosophy.

. . .

The law of thought expressed by the equation (1) will . . . be occasionally referred to as the "law of duality."

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Task 12

In outline form, Boole's proof of Proposition IV is as follows:

$$\begin{aligned} \text{Given:} \quad & x = x^2. \\ \text{Then} \quad & x - x^2 = 0, \\ \text{and} \quad & x(1 - x) = 0 \end{aligned}$$

Interpreting the equation $x(1 - x) = 0$, we conclude that the classes x and $1 - x$ have no common members. Thus, it is impossible for "any being to possess a quality, and at the same time not to possess it."

In his discussion of Proposition IV, Boole made the following additional comments (emphasis added):

The above interpretation has been introduced not on account of its immediate value in the present system, but *as an illustration of a significant fact* in the philosophy of the intellectual powers, viz., *that what has been commonly regarded as the fundamental axiom of metaphysics is but the consequence of a law of thought, mathematical in its form.*

Does Boole's formal manipulation of the law $x^2 = x$ provide convincing proof that the *Principle of Contradiction* is a 'consequence of a law of thought, mathematical in its form,' rather than being an 'axiomatic law'? Explain.

Task 13

In a footnote to Proposition IV (omitted here), Boole discussed the status of the third degree equation $x^3 = x$ in his system, and remarked that this equation can be factored as follows:

$$x(1-x)(1+x) = 0$$
$$x(1-x)(-1-x) = 0$$

He then concluded that $x^3 = x$ is 'not interpretable in the system of logic' because both factorizations contain meaningless factors, $1 + x$ and $-1 - x$ respectively. Comment on why these two expressions ($1 + x$ and $-1 - x$) are meaningless in Boole's system. [Hints? Consider (a) Boole's definition of addition, and (b) the symbol -1 relative to the second degree law $x^2 = x$.] Then provide an argument *for* considering the equation $x^3 = x$ to be a valid law of symbolic logic.

Having established the basic algebraic laws his symbolic representational system for classes in Chapters I–III of his *Laws of Thought*, Boole turned in Chapter IV to the issue of symbolically representing 'the complex terms of propositions' in that system, again with the overall goal of deducing new propositions from a set of given propositions simply through algebraic manipulation of the associated symbolic expressions. Here, a 'proposition' is understood to be a statement which has a definite (but possibly unknown) truth value. For example, the statement 'My neighbor has a cat.' is a proposition since it must be either true or false, even though we may not know which value pertains. In contrast, neither the question 'Does my neighbor have a cat?' nor the exclamation 'Hi neighbor!' is a proposition. Boole described several rules for expressing propositions in his symbolic language for classes, the first of which is given below [4, p. 57]:

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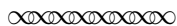
RULE- Express simple names or qualities by the symbols x, y, z &c, their contraries by $1-x$, $1 - y$, $1 - z$, &c.; classes of things defined by common names or qualities, by connection the corresponding symbols as in multiplication ; collection of things consisting of portions different from each other, by connecting the expression of those portions by the sign $+$. In particular, let the expression, "Either x's or y's," be expressed by $x(1-y)+y(1-x)$, when the classes denoted by x and y are exclusive, by $x + y(1 - x)$ when they are not exclusive. Similarly let the expression "Either x's or y's, or z's," be expressed by $x(1 - y)(1 - z) + y(1 - x)(1 - z) + z(1 - x)(1 - y)$, when the classes denoted by x, y and z are exclusive, by $x + y(1 - x) + z(1 - x)(1 - y)$ when they are not meant to be exclusive, and so on.

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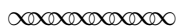
Task 14

Boole then gives several examples of how to use this rule, the first of which involves the following classes: x represents hard substances, y represents elastic substances, and z represents metals. Use Boole's rules to express each of the following in these terms; remember Boole's restriction on exception!

(a) non-elastic metals

(b) hard substances, except metals

(c) hard substances, except those which are metallic and non-elastic and those which are elastic and non-metallic.

(d) hard substances except those which are metallic and non-elastic, and substances which are elastic and non-metallic.

(e) metallic substances, except those which are neither hard nor elastic.

Elsewhere in Chapter IV [4, p. 57], Boole discussed the 'mutually exclusive' issue further:

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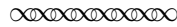
And thus, according to the meaning implied, the expression "Things which are either x's or y's," will have two different symbolical equivalents. If we mean, "Things which are x's, but not y's, or y's, but not x's," the expression will be
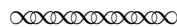
$$x(1 - y) + y(1 - x);$$

the symbols x standing for x's, y for y's. If, however, we mean "Things which are x's, or, if not x's, then y's," the expression will be

$$x + y(1 - x).$$

This expression supposes the admissibility of things which are both x's and y's at the same time. It might more fully be expressed in the form

$$xy + x(1 - y) + y(1 - x);$$

but this expression, on addition of the two first terms, only reproduces the former one.

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Task 15
In other words, Boole is telling us that $x + y(1 - x)$ and $xy + x(1 - y) + y(1 - x)$ are *both* legitimate ways to express a non-exclusive 'or' for two classes x and y, since $x = xy + x(1 - y)$.

For three classes x, y and z, Boole's rule (above) indicates that one way to express things that are "Either x's or y's, or z's" under the non-exclusive interpretation is

$$x + y(1 - x) + z(1 - x)(1 - y).$$

Expand the first two terms of this expression (in a fashion similar to Boole's expansion for two classes) in order to obtain an expression involving 7 distinct terms which represents the class of things that are "either x's or y's, or z's" under the non-exclusive interpretation.

3 Venn: Refinements of Boole's Symbolic Logic

Although few copies of Boole's *Laws of Thought* were sold when it first appeared in 1854, the algebraic methods introduced by Boole for the study of logic did attract the attention of mathematicians in ensuing years. In the next two sections of this project, we consider refinements and extensions of Boole's system made by two of these individuals. We begin with excerpts from the second (1894) edition of *Symbolic Logic* by John Venn (1834 – 1923).

Venn's father and grandfather were both priests of the Church of England, as well as active leaders in the evangelic Christian movement. His grandfather's sect was involved in various progressive social movements, such as prison reform and the abolition of slavery; his father was active in missionary efforts in Africa and Asia. As was expected of him, Venn entered the priesthood, after first completing a degree in mathematics at Cambridge in 1857. Unlike his forebears, he then entered academia and assumed the position of lecturer in Moral Science at Cambridge in 1862. He married in 1867 and, much later, collaborated on an extensive history of Cambridge with his only son, also an academic. History was, in fact, Venn's focus in the latter part of his career; his writings on logic also included extensive historical notes. His early interests lay primarily in philosophy, logic and probability, however. Venn's publications in logic appeared primarily in the 1880's. Always a man of conviction, Venn left the priesthood around this same time due to differences between his own beliefs and those of the evangelical creed.

Venn's name is best known today in connection with the overlapping circle diagrams commonly used to represent the operations of 'intersection' and 'union' on sets; in essence, these set operations correspond to Boole's operations of addition and multiplication on classes. Venn himself called these diagrams 'Eulerian Diagrams' in recognition of their earliest known use as a means to represent categorical propositions (e.g., *All S are P* , *Some S are not P*) by Leonhard Euler (1707–1783). In Venn's work, these diagrams were used to represent both categorical propositions and operations on classes.[4]

To see how categorical propositions and Boole's symbolic algebra for classes are related, consider the following excerpt from Venn's *Symbolic Logic* [24, pp. 26-27]:

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Writing, for simplicity, $\overline{x}$ for not-x and $\overline{y}$ for not-y, ... :

$$xy = 0, \quad \text{or} \quad \text{No } x \text{ is } y.$$
$$\overline{x}y = 0, \quad \text{or} \quad \text{All } y \text{ is } x.$$
$$x\overline{y} = 0, \quad \text{or} \quad \text{All } x \text{ is } y.$$
$$\overline{x}\overline{y} = 0, \quad \text{or} \quad \text{Everything is either } x \text{ or } y$$

On this plan of notation xy stands for the compartment, or class, of things which are both x and y; and the equation $xy = 0$ expresses the fact that that compartment is unoccupied, that there is no such class of things.

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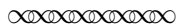
[4]To represent operation on classes, Venn would actually shade those regions of the diagram which were excluded from the class in question. In modern usage, the regions included in the class are shaded. Refer to a modern text on discrete mathematics or set theory for examples.

Notice Venn's use of the notation $\overline{x}$ to represent the class of things which have the property not-x in this excerpt. Elsewhere in *Symbolic Logic*, he simply wrote 'not-x' to designate it, or used Boole's notation $1-x$ to represent this same class. Venn also warned practitioners of a certain danger that arises with regards to the use of the symbol $-A$ as notation for this class [24, pp. 56-57]:

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This use of the negative sign must not be confounded with another use which has found favour from time to time, viz. that of employing it to designation negative terms. . . . Thus 'man' and 'not-man', taken together, should comprise 'all', i.e. the logical universe; but $+A$ and $-A$ taken together neutralize each other and result in 0.

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In other words, Venn was warning his readers that the equation "$A + (-A) = 0$" *is* correct as an algebraic statement, but that it is *not* correct as a logical statement. Thus, to avoid potential errors, the expression '$-A$' should *not* be used to represent the class 'not-A'.

Task 16

Let x to represent the class 'men,' and consider the correct logical principle which Venn described as follows:

" 'man' and 'not-man', taken together, should comprise 'all', i.e. the logical universe

First represent this logical principle as an equation using Venn's notation '$\overline{x}$' to designate the class 'not-men.' Then represent this same principle as an equation using Boole's notation '$1-x$' to designate the class 'not-men.' Comment on the advantages and disadvantages of these two methods of representation.

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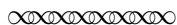
Although Venn wrote his text primarily from the perspective of a logician, his treatment of Boole's work addressed several details of direct concern to mathematicians. The first of these related to the nature of the operation $+$, which Boole insisted on using in an exclusive sense. Here is Venn's writing on this subject [24, pp. 42-46]:

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. . . we often require to group two or more classes together, so as to make one aggregate class of them. . . . There ought not, one would think, to be much opening here for doubt and confusion. What however with the ambiguities of popular language, and the disputes of rival modes of symbolic statement, a little cloud of confusion has been stirred up. The possibility of this has arisen mainly from a want of proper care in clearly distinguishing between the various questions at issue.

(1) To begin with, there being members common to two classes, we may have it in view to *exclude* these from our aggregate. In physical problems, it may happen that one or other alone of two causes will produce a certain effect, but that the two together will either

neutralize each other, or by their excess produce some quite different effect. Or, to take a familiar example of another kind, we might have it in view to announce the pardon of two classes of offenders, but expressly wish to exclude the aggravated cases which fall under both heads.

(2) The more frequent case, however, is that in which the common members are included in our formula, so to say, by a double right. ...

(3) But there is still a third possible case for examination. May we ever want to reckon this common part *twice over?* In numerical application of our formula, or whenever consideration of quantity are in any way introduced by them, it seems to me quite clear that we may have to do so. ...Suppose, for instance, that all poachers and trespassers are to be fined 20 shillings: is it quite certain that poachers who trespass, could not be fined 40 shillings? This is, I apprehend, a question for the lawyers to decide. But the language is, to common understanding, certainly ambiguous, which is sufficient for our present purpose. ...

There being, then, these three possible modes of class aggregation, the question arises how they should be respectively indicated. If all three were really in frequent use, there would be very little doubt, I think, that we should have to mark them as follows

$$A \text{ not-}B + B \text{ not-}A, \quad A + B \text{ not-}A, \quad A + B.$$

In the first we exclude the AB members; in the second we simply count them; in the third we count them twice.

As a matter of fact, however, the third case belongs to Applied rather than to Pure Logic ...As, however, such examples are very exceptional, we may lay them aside, and consider that we have, in Pure Logic, to deal with the former methods only of aggregation.

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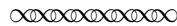
Task 17
Before continuing your reading of Venn, consider how these three modes of class aggregation fit with Boole's use of '+'. Explain why the first two modes ('A not-$B + B$ not-A' and '$A+B$ not-A') are always meaningful under Boole's restriction on the symbol "+", while the third ('$A + B$') is sometimes but not always meaningful under Boole's restriction. Provide a specific example of classes A and B in which 'A not-$B + B$ not-A' and '$A + B$ not-A' represent different classes. Is the expression '$A + B$' meaningful in Boole's system in this example? Explain.

Note: Today, the mode of class aggregation which Venn denoted by 'A not-$B+B$ not-A' is referred to as the *symmetric difference*.

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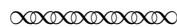
The question then becomes simply one of procedure and notation. When we want to represent the aggregate class 'A and B,' shall we expressly exclude double counting by writing it $A + B$ not-A, or shall we write it $A + B$, with the proviso, made once for all, that this shall not be considered as counting AB twice over? As just remarked, the question is one of procedure only, but of procedure in a very wide sense; ...at the same time, it must be remembered, the actual original difference of opinion is very small. All alike agree, I

apprehend, that classes must as a matter of fact admit of being aggregated in the ways just described. Moreover all alike agree that when we want to express 'X or Y, but not both,' we mark this distinctively. The only difference of opinion is in the treatment of 'X or Y or both':— shall we formally exclude double counting by our notation, or shall we assume as a general convention that is it not to be admitted?

Boole, as is well known, adopted the former plan, making all his alternatives mutually exclusive, and in the first edition of this work I followed his plan. I shall now adopt the other, or non-exclusive notation:– partly, I must admit, because the voting has gone this way, and in a matter of procedure there are reasons for not standing out against such a verdict; but more from a fuller recognition of the practical advantages of such a notation. . . . as a rule and without intimation of the contrary, I shall express 'X or Y,' in its ordinary sense, by $X + Y$. I regard it as a somewhat looser mode of statement, but as possessing, amongst other advantages, that of very great economy.

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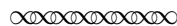
Task 18

To illustrate the 'very great economy' to be gained by Venn's intepretation of $+$ in comparison to that of Boole, consider the following example taken from Chapter IV of Boole's *Laws of Thought* [5, p. 59].

$$
\begin{aligned}
w &= \text{wealth} \\
t &= \text{things transferable} \\
s &= \text{limited in supply} \\
p &= \text{productive of pleasure} \\
r &= \text{preventative of pain}
\end{aligned}
$$

Here Boole wished to represent the following definition of 'wealth,' proposed by Victorian economist Nassau William Senior (1790–1864), in symbolic form: *Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventative of pain.*

In order to represent the class of things which are "either productive of pleasure or preventative of pain" as the sum of *disjoint* classes, Boole was required to write $p + (1 - p)r$. Thus, he translates Senior's definition of wealth into the symbolic expression $w = ts\{p + (1 - p)r\}$.

In contrast, under Venn's interpretation of $+$, Senior's definition can be represented symbolically as $w = ts(p + r)$.

 (a) Explain why the class which Boole expressed by $p + (1 - p)r$ does include those things which are *both* productive of pleasure and preventative of pain. How would Boole represent this same class under the interpretation that things which are both productive of pleasure and preventative of pain are to be excluded? How would Venn have written this class under this same interpretation?

 (b) Compare the symbolic expressions for the two classes listed in (i) and (ii) below under both two interpretations of $+$ (inclusive versus inclusive), first in Boole's system and then in Venn's system. Use the notation $1 - x$ to represent the class of not-x in both cases, and illustrate the sets in question by a Venn diagram.

    (i) things transferable or not limited in supply

    (ii) things either not transferable or not productive of pleasure

*[Note that both parts (i) and (ii) require **four** answers each: the indicated set as it would be written using (α) Boole's '+' under an inclusive interpretation; (β) Venn's '+' under an inclusive interpretation; (γ) Boole's '+' under an exclusive interpretation; and (δ) Venn's '+' under an exclusive interpretation. It may be helpful to arrange these in a 2×2 table.]*

(c) Comment on the advantages of Venn's system relative to those of Boole's system in terms of economy or efficiency of expression.

    A second mathematical issue addressed by Venn pertained to expressions of the form $x - y$, which Boole considered meaningless unless y was included in x. On this count, Venn agreed with Boole for the following reasons [24, pp. 49-56]:

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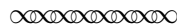
It is often convenient to begin by taking account of some aggregate class, and then to set aside a portion of it and omit this from consideration. This process is the simple inverse of that discussed above *[e.g., aggregation]*, and some of the difficulties to which its expression has given rise spring from corresponding causes. Three cases may be noticed in order.

(1) The class to be excepted may be included within the other . . . In this case the omission or exclusion is perfectly simple and intelligible.

(2) Again, if the classes are mutually exclusive, the omission of either from the other is equally simple, but unintelligible. The direction to omit the women from the men in a given assembly has no meaning. At least it will require some discussion about the interpretation of symbolic language in order to assign a meaning to it.

(3) But if two classes are partly inclusive and partly exclusive of one another, we are landed in somewhat of a difficulty. What, for instance, would be meant by speaking of "all trespassers, omitting poachers"? If we interpreted this with the rigid stringency with which we treat algebraical symbols, we should have to begin by deducting the common part, viz. the poachers who trespass, from the trespassers; and should then be left with an unmanageable remainder, viz. the poacher who did not trespass, and who could not therefore be omitted.

Popular language is of course intended to provide for popular wants, and must be interpreted in accordance with them. . . . Accordingly when we meet with such a phrase as that in question, we take it for granted that any such uninterpretable remainder is to be disregarded, and we understand that 'trespassers, omitting poachers' is to be taken to signify 'trespassers, omitting such trespassers as poach'.

<p style="text-align:center">. . .</p>

The deduction, omission, or subtraction, . . . of one class from another is expressed . . . by aid of the symbol $(-)$.

<p style="text-align:center">. . .</p>

In using this symbol we must remember the condition necessarily implied in the performance of the operation which it represents. As was remarked, we cannot 'except' anything from that in which it was not included; so that $x - y$ certainly implies that y is a part of x. This is quite in accordance with the generalized use of the symbol in mathematics where it is

<p style="text-align:center">19</p>

always considered to mark the undoing of something which had been done before. Not of course that it must thus refer to the immediately preceding step, but that there must be some step or steps, in the group of which it makes a part, which it can be regarded as simply reversing. The expression $x - y + z$ will be satisfactory, provided either x or z is known to be inclusive of y, or if both combine together to include it.

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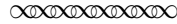
Task 19

Comment on the two arguments Venn provided for restricting the use of the symbol '$-$' in logic, the first based on popular language usage ['we take it for granted that any uninterpretable remainder is to be disregarded'], and the second based on analogy with algebra ['it is always considered the mark of the undoing of something which had been done before']. Discuss in particular what is being 'undone' in the second argument. How convincing do you find these arguments?

Interestingly, after arguing that subtraction is acceptable in symbolic logic subject to the restriction discussed above, Venn went on to prove that this operation can be omitted from the system of symbolic logic altogether. His argument in this regard employed the definition of multiplication for classes which was given by Boole: '$x \times y$, or xy, will stand for the things which are both x and y [24, p. 59].' Using class multiplication, Venn then argued that subtraction may be omitted from the system of symbolic logic for classes as follows [24, pp. 68-69]:

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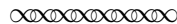
It will now be seen how it is that the process of subtraction, with the corresponding symbolic sign, can be dispensed with. If y be excepted from x it must be a part of x, and may therefore be written xy .... But this, as we have just explained, is the same as to 'multiply' x by $1 - y$, or by not-y. In other words, the legitimate exception of y from x is the same thing, in respect of the result, as taking the common part of x and not-y. 'The clergy, except the teetotalers', means, when the exception is duly interpreted, the class common to those of 'the clergy' and 'the non-teetotalers'.

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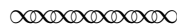
Task 20

In other words, by multiplying the classes 'x' and '$1 - y$' defined in this excerpt, Venn was then able to represent the class 'clergy, except the teetotalers' by the expression '$x(1-y)$'. Yet Venn intended this example to serve as an illustration of how 'the process of subtraction, with the corresponding symbolic sign, can be dispensed with.' Explain why there is no inconsistency here. Then indicate how the class 'The clergy, except the teetotalers' could be represented symbolically by Venn *without* using the symbol '$-$' for any purpose.

Although the operation of logical subtraction 'can certainly be dispensed with,' Venn goes on to say that 'the notion however of this process of exception, in a sense analogous to (but not identical with) that of arithmetical subtraction, is far too familiar to popular thought for us to be able to ignore its existence, though we shall not very often have occasion to employ it in practice' [24, p. 69]. The one possible occasion for its employment discussed in detail by Venn was the following [24, pp. 58-59]:

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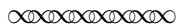
The question may naturally be asked whether we may really follow the mathematical analogy and transfer a term from one side of an equation to the other, with due change of sign. ...Start with the case of a *sum* of terms, viz. with terms connected by the sign $+$. Here we can simply transfer a term if the alternatives are, either formally or materially, exclusive, but not otherwise. Thus, barristers and solicitors comprising the lawyers[5], the lawyers omitting the solicitors comprise the barristers ($x + y = z$; $\therefore z - x = y$). ...But if the two classes overlap each other, we should then find that, in making the transfer, we were carrying this common part over: unless of course we are careful to make a corresponding correction. ...The power of free subtraction without the need of any such correction is one of the conveniences of the Boolian plan of notation.

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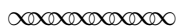
Task 21
What correction is needed to transfer y to the other side in the equation $x + y = z$ in the case of two overlapping classes? That is, given $x + y = z$ where $xy \neq 0$, what is the legitimate symbolic expression for y in Venn's system? Sketch a Venn diagram to illustrate your reasoning. (It may also be helpful to consider a concrete example involving overlapping classes, such as those considered in Task 7.) Comment on why 'the Boolian plan of notation' does not require such a correction.

Task 22
Will any correction be needed in the case where we start with a difference instead of a sum? That is, if $z = x - y$ in Venn's system, can we always conclude that $z + y = x$? Why or why not? Sketch a Venn diagram to illustrate your reasoning, and compare this to the situation in Boole's system. (Remember the restrictions which Boole and Venn placed upon subtraction.)

We conclude our reading of Venn's work with an excerpt on 'multiplication' [24, pp. 69-70]:

---

[5]In the United Kingdom and Ireland, the legal profession is split between solicitors and barristers, and a practitioner of the law will usually only hold one title.

While on the subject of 'multiplication' two very useful formulæ may be noticed. Most of the cases which occur in practice are so simple that the beginner will have no difficulty in carrying them out. There are, however, some cases of such frequent occurrence that it is well worth while to keep in mind a few simple rules for the abbreviation of a process which is often apt to prove tedious.

(1) $(A + x)(B + x) = AB + x.$ . . . . . .

(2) $(A + x)(B + \overline{x}) = A\overline{x} + Bx.$ . . . . . .

The former of these rules is given, as a special method of simplification, by Boole (p. 132). The latter, with the full significance of these abbreviations, is due to Professor Peirce.

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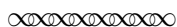
Task 23

Venn's proofs for why these laws hold has been omitted from the excerpt above.

(a) Provide an argument for why these two formulæ must hold by considering the specific concrete example of classes A , B and x, where A represents acrobats, B represents ballerinas, and x represents xylophonists, and describing the members of the classes represented by the expressions on either side of the equal sign.

(b) Reconstruct Venn's proofs of each formula by algebraically expanding the left-hand side to obtain four terms, and then explaining why two of those four terms can be ignored.

(c) Compare these two methods of justification (e.g., concrete interpretation versus formal manipulation) in terms of their rigor and ease of use.

Task 24

As noted by Venn, Boole included the first of these two formulæ in his *Laws of Thought* (in a chapter we did not consider in this project). What conditions on A, B and x would be needed in order for the second formula to be acceptable in Boole's system? Sketch a Venn diagram illustrating classes A, B and x which satisfy these conditions. Then explain why this second formula can be simplified to $(A + x)(B + \overline{x}) = A + B$ when these conditions are met.

4   Peirce: Increasing Levels of Abstraction

The first edition of Venn's *Symbolic Logic* was published in 1881; the excerpts used in this project were drawn from the second edition, published in 1894. In his preface to the second edition, Venn remarked that Boole's ideas had become increasingly acceptable in the interim [24]:

At the time when the first edition of this work was composed it would scarcely be too much to say that the conception of a Symbolic Logic was either novel or repugnant to every professional logician. An amount of explanation and justification was therefore called for which does not seem to be quite so necessary now.

In his introduction, Venn also suggested a possible cause for the early prejudice against Boole's mathematical approach to logic [24, pp. ix-x]:

Those generalizations to which Boole was the first to direct attention, and which ... are purely logical in their origin and nature, are yet of a character which it is very difficult for any one to grasp who has not acquired some familiarity with mathematical formulæ. This should not be so, and I hope will not always continue to be so, but at present it is almost inevitable. The state of things arises from the fact that we have to realise a certain generalized use of symbols; that is, to take account of an extension of their use to cases distinct from, but analogous to, that with reference to which they first had their meanings assigned. ... This deliberate and intentional transfer of signification to analogous cases, as it happens, is perfectly familiar to the mathematician, but very little recognized or understood in other departments of thought.

In fact, several mathematicians had already generalized Boole's ideas beyond his basic approach to logic during the interim between the appearance of *Laws of Thought* in 1857 and *Symbolic Logic* in 1881. One of the most historically important of these in terms of later developments was the American mathematician Charles Sanders Peirce (1839–1914)[6].

Peirce's father Benjamin, an algebraist, was one of the earliest American research mathematicians; an older son of Benjamin also worked in mathematics. All four Peirce sons were schooled at home by their father; by the time he was 12, Charles was reading university level texts on logic. He entered Harvard at the age of 16, completing both an undergraduate and a master's degree; in 1863, he was awarded a master's degree in chemistry by the Lawrence Scientific School. Regrettably, both his personal and professional life were marked by turbulence. He suffered from a painful nervous and facial condition for most of his life, and was described by many as having a difficult personality. He was employed in various scientific capacities by the US Coast Guard until 1891, but appeared able to maintain this connection largely because of his father's influence. In 1876, he separated from his first wife after thirteen years of marriage, but only became officially divorced another seven years later. Having lived with the woman who would become his second wife in the interim, Peirce was dismissed from his position as professor of logic at Johns Hopkins in 1883 to avoid a scandal. He owed this short-lived academic position to fellow British mathematician J. J. Sylvester (1814–1897), himself a colorful character. Other 'colleagues' appear to have actively worked to block Peirce from an academic career. Supported in part by loans and donations from friends and in part by meager pay from various writing jobs, Peirce lived the latter years of his life in poverty. Throughout this time, he continued his work in logic and philosophy. Destitute at the time of his death, from cancer, his second wife survived him by some twenty years.

Peirce's extensive contributions to philosophy reached beyond logic, and included epistemology, metaphysics, pragmatism and the philosophy of science. One of the notable contributions which he made to the development of symbolic logic was a systematic study of the relation 'inclusion' for

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[6]The name 'Peirce' is pronounced the same as the word 'purse.'

classes and the corresponding relation 'material implication' in propositional logic[7]. The concept of 'class inclusion' was quickly adopted from Peirce by the German mathematician Schröder (1841–1902) as the centerpiece in his development of symbolic logic; Schröder's monumental three volume work *Vorlesungen über die Algebra der Logik* was in turn highly influential on the thinking of later mathematical logicians.

We focus here on Peirce's earliest published work on symbolic logic, his *On an Improvement in Boole's Calculus of Logic* of 1867, written prior to his study of the relation of class inclusion. Interestingly, Peirce focused this paper on the application of Boole's system to probability calculations, rather than its application to logic. Boole's own interest in this connection is evident from the full title of his *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, in the latter third of which Boole considered the foundations of probability based on his algebra of logic. Peirce described the connection between this algebra and probability in his opening paragraph [17, p. 3]:

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The principal use of Boole's Calculus of Logic lies in its application to problems concerning probability. It consists, essentially, in a system of signs to denote the logical relations of classes. The data of any problem may be expressed by means of these signs, if the letters of the alphabet are allowed to stand for the classes themselves. From such expressions, by means of certain rules for transformation, expressions can be obtained for the classes (of events or things) whose frequency is sought in terms of those whose frequency is known. Lastly, if certain relations are known between the logical relations and arithmetical operations, those expression for events can be converted into expressions for their probability.

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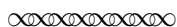
Note Peirce's emphasis on the distinction between 'logical relations' and 'arithmetical operations' above. This distinction is further emphasized in his definitions of equality and addition as they pertain to classes [17, pp. 3-4]:

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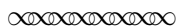
Let the letters of the alphabet denote classes whether of things or of occurrences. ... Let the sign of equality with a comma beneath it express identity. Thus $a \underset{,}{=} b$ is to mean that a and b denote the same class — the same collection of individuals.

Let $a +_, b$ denote all the individuals contained under a and b together. The operation here performed will differ from arithmetical addition in two respects: first, that it has reference to identity, not to equality; and second, that what is common to a and b is not taken into account twice over, as it would be in arithmetic. The first of these differences, however, amounts to nothing, inasmuch as the sign of identity would indicate the distinction in which it is founded; and therefore we may say that

(1)   If No a is b $a +_, b \underset{,}{=} a + b$

It is plain that

(2) $$a \mathbin{\tilde{+}} a \mathbin{\tilde{=}} a$$
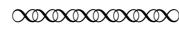
and also, that the process denoted by $\tilde{+}$, and which I shall call the process of *logical addition*, is both commutative and associative. That is to say

(3) $$a \mathbin{\tilde{+}} b \mathbin{\tilde{=}} b \mathbin{\tilde{+}} a$$

and

(4) $$(a \mathbin{\tilde{+}} b) \mathbin{\tilde{+}} c \mathbin{\tilde{=}} a \mathbin{\tilde{+}} (b \mathbin{\tilde{+}} c)$$

∞∞∞∞∞∞∞∞∞∞

Task 25
Speculate on why Peirce considered it important to use modified symbols ($\tilde{=}$, $\tilde{+}$) when referring to operations on classes, rather than the standard mathematical symbols which Boole employed ($=$, $+$). Does this make it easier or harder to:

 (a) remember that the operations are on classes (e.g., sets) rather than numbers?

 (b) remember the analogies between the two systems?

 (c) follow Peirce's writing?

Might these three goals be better served by introducing entirely new symbols, rather then simply modifying existing symbols? Explain.

Task 26
Under Boole's disjoint (exclusive) interpretation of addition, Peirce's identity (2) is meaningless. Under the non-disjoint (inclusive) interpretation of addition adopted by Peirce and Venn, is it plain this identity holds for classes? Explain.

Task 27
Later in his paper, Peirce defined two special elements, 0 and 1, which played analogous roles in his algebra to those played by the elements which Boole and Venn denoted by these same symbols in their systems. When applied to these two particular elements, Peirce's identity (2) implies that $0 + 0 = 0$ and $1 + 1 = 1$. Letting '0' represent the probability of an impossible event and '1' represent the probability of a certain event, do these two (numerical) equations make sense in the context of probability computations? For example, if A and B are both impossible events, what probability should be assigned to the event represented by $A \mathbin{\tilde{+}} B$, and why? What if A and B are both certain events?

Peirce also used the analogy of 'logical calculations' and probability computations to justify using the term 'multiplication' for the class operation of selecting common elements [17, p. 4]:

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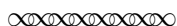
Let a,b denote the individuals contained at once under the classes a and b; those of which a and b are the common species. If a and b were independent events[8], a, b would denote the event whose probability is the product of the probabilities of each. On the strength of this analogy (to speak of no other), the operation indicated by the comma may be called *logical multiplication*.

It is plain that

(5) $$a,a \mathrel{\overline{\overline{}}} a.$$

Logical multiplication is evidently a commutative and associative process. That is,

(6) $$a,b \mathrel{\overline{\overline{}}} b,a$$

and

(7) $$(a,b),c \mathrel{\overline{\overline{}}} a,(b,c).$$

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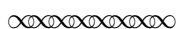Before continuing your reading of Peirce, take note of the symmetry between the identities numbered (2) and (5) , those numbered (3) and (6) , and those numbered (4) and (7) respectively. This same 'duality' — in which logical addition replaces logical multiplication and vice versa — is also visible in his next two identities on distributivity. Unlike his first six identities, which were put forward as 'evident' based solely on the assigned meanings of the operations, Peirce *deduced* identities (8) and (9) as formal theorems. The excerpt below includes his proof of identity (8); we will examine this proof in detail in Task 28.

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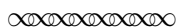
Logical addition and logical multiplication are doubly distributive, so that

(8) $$(a \mathbin{+\!\!\!+} b),c \mathrel{\overline{\overline{}}} a,c \mathbin{+\!\!\!+} b,c$$

and

(9) $$(a,b) \mathbin{+\!\!\!+} c \mathrel{\overline{\overline{}}} (a \mathbin{+\!\!\!+} c),(b \mathbin{+\!\!\!+} c).$$

Proof.   Let

$$\begin{aligned} a &\mathrel{\overline{\overline{}}} a' + x + y + o \\ b &\mathrel{\overline{\overline{}}} b' + x + z + o \\ c &\mathrel{\overline{\overline{}}} c' + y + z + o \end{aligned}$$

where any of these letters may vanish. These forumulæ comprehend every possible relation of a , b and c; and it follows from them that

$$a \mathbin{+\!\!\!+} b \mathrel{\overline{\overline{}}} a' + b' + x + y + z + o \qquad\qquad (a \mathbin{+\!\!\!+} b),c \mathrel{\overline{\overline{}}} y + z + o$$

But

$$a,c \mathrel{\overline{\overline{}}} y + o \qquad b,c \mathrel{\overline{\overline{}}} z + o \qquad a,c \mathbin{+\!\!\!+} b,c \mathrel{\overline{\overline{}}} y + z + o \qquad \therefore (8)$$

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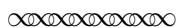
[8] As an example of *independent events*, consider the random experiment of tossing a single fair coin twice in succession, where the outcome of the first toss does not effect that of the second; thus, the probability of obtaining heads on a second toss is $\frac{1}{2}$, regardless of the outcome of the first toss, and the probability two heads in a row is given by $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. In contrast, if two cards are drawn from a standard deck without replacement, the the probability that the second card is a heart depends on the outcome of the first draw; such events are said to be *dependent*.

<div align="center">26</div>

Task 28

In his proof of identity (8), Peirce asserted that the following formulæ "comprehend every possible relation of a , b and c."

$$
\begin{aligned}
a &\overset{.}{=} a' + x + y + o \\
b &\overset{.}{=} b' + x + z + o \\
c &\overset{.}{=} c' + y + z + o
\end{aligned}
$$

(a) Examine these formulas carefully to see how the first one could be re-written in Venn's notation as follows[9]:

$$
a = a(1-b)(1-c) + ab(1-c) + ac(1-b) + abc
$$

Then complete the following table for the terms in the formula representing a:

Class	Venn's Representation	Verbal Description of Members
a'	$a(1-b)(1-c)$	a but not-b and not-c
x		
y		
o		

Use this table to explain the sense in which the formula $a \overset{.}{=} a' + x + y + o$ 'comprehends every possible relation of a with b and c'.

(b) Now return to Peirce's notation for all three formulæ,

$$
a \overset{.}{=} a' + x + y + o \qquad b \overset{.}{=} b' + x + z + o \qquad c \overset{.}{=} c' + y + z + o,
$$

and recall that Peirce's logical addition and logical multiplication correspond to Venn's addition (e.g., inclusive aggregation) and multiplication (e.g., common elements) respectively. Explain how these definitions and the given formulæ imply the following two equations must hold:

$$
a +\!\!,\, b \overset{.}{=} a' + b' + x + y + z + o \qquad\qquad (a +\!\!,\, b), c \overset{.}{=} y + z + o
$$

(c) Similarly, explain how Peirce's definitions of logical addition and logical multiplication imply the following three equations must hold:

$$
a, c \overset{.}{=} y + o \qquad b, c \overset{.}{=} z + o \qquad a, c +\!\!,\, b, c \overset{.}{=} y + z + o
$$

(d) Complete the proof of property (8) by comparing the expressions obtained above for $(a +\!\!,\, b), c$ and $a, c +\!\!,\, b, c \overset{.}{=} y + z + o$.

---

[9]Because the terms on the right-hand side of each equation represent disjoint classes, these equations would also be meaningful in Boole's system. The disjointness of these classes, together with Peirce's property (1), also allowed Peirce to simply use the symbol '+' on the right hand side of these formulæ, rather than his usual symbol '+,'.

Task 29

Now consider Peirce's identity (9) for the distributivity of (logical) addition over (logical) multiplication. Recall that Venn wrote this as $(A + x)(B + x) = AB + x$, a statement which you proved in project question 23 (using the distributivity of multiplication over addition to expand $(A+x)(B+x)$ and then simplifying). Provide a proof of this identity following the format of Peirce's proof for identity (8), and using Peirce's notation, definitions and the formulae

$$a \underset{\cdot}{=} a' + x + y + o \qquad b \underset{\cdot}{=} b' + x + z + o \qquad c \underset{\cdot}{=} c' + y + z + o.$$

We conclude our reading of Peirce with one final excerpt on the operation of 'logical subtraction' [17, p. 5]:

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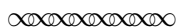
Let $\underset{,}{-}$ be the sign of logical subtraction; so defined that

(10) $\qquad\qquad$ If $b \underset{,}{+} x \underset{\cdot}{=} a \qquad\qquad x \underset{\cdot}{=} a \underset{,}{-} b.$

Here it will be observed that x is not completely determinate. It may vary from a to a with b taken away. This minimum may be denoted by $a - b$. It is also to be observed that if the sphere of b reaches at all beyond a, the expression is uninterpretable.

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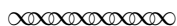
Task 30

In this question, we explore why Peirce's definition of logical subtraction as the formal inverse of logical addition leads to an operation which is 'not completely determinate' using a specific example of finite classes. To this end, let $a = \{1, 2, 3, 4, 5, 6, 7\}$ and $b = \{2, 4, 6\}$.

(a) Let $y = \{1, 2, 3, 5, 7\}$ and let z represent 'a with b taken away,' so that $z = \{1, 3, 5, 7\}$.

Verify that $b \underset{,}{+} y \underset{\cdot}{=} a$ and that $b \underset{,}{+} z \underset{\cdot}{=} a$.

Explain why this allows us to conclude that $y = \{1, 2, 3, 5, 7\}$ is *one* of the possible classes which *could* be assigned to $a \underset{,}{-} b$ and that $z = \{1, 3, 5, 7\}$ is another such possibility.

(b) Use an addition statement to explain why a is *another* possible class which *could* be assigned to $a \underset{,}{-} b$ under Peirce's formal definition.

(c) Explain, in general, why *any* class which minimally includes all members of the class 'a with b taken away' without 'reaching beyond a' could also be used as a value for $a \underset{,}{-} b$.

(d) Although the class $a \underset{,}{-} b$ is 'not completely determinate,' Peirce does define the class $a - b$ (without the comma subscript) as a definite well-determined class, namely the minimum value of x for which $b \underset{,}{+} x \underset{\cdot}{=} a$. For the particular example considered in this question, determine the class represented by the expression $a - b$. Then explain why, in general, the class $a - b$ defined by Peirce is necessarily disjoint from the class b (assuming 'the sphere of b' does not reach beyond a, so that the expression is interpretable.)

Task 31

Use Peirce's definition of logical subtraction as the formal inverse of logical addition to explain why the expression $a \mathbin{-_{\scriptscriptstyle\backslash}} b$ is uninterpretable if 'the sphere of b reaches at all beyond a'. That is, if b is not included in a, what part of the definition of $a \mathbin{-_{\scriptscriptstyle\backslash}} b$ fails to hold, and why? Compare this rationale for restricting $a \mathbin{-_{\scriptscriptstyle\backslash}} b$ to cases where b is included in a to those offered by Boole and Venn (course questions 7 and 19).

Task 32

Beginning in the twentieth century, it became standard for mathematicians working in symbolic logic (or boolean algebra) to limit their attention to just two primary boolean algebra operations analogous to Peirce's logical addition and logical multiplication. Write a paragraph discussing this mathematical development in light of what we have seen about subtraction in the work of Boole, Venn and Peirce. (Project questions 7, 8, 19, 20, 21, 22, 30 and 31 relate to this issue.)

Also write a paragraph speculating on why the operation of division does not arise in a modern Boolean Algebra. (In fact, Boole, Venn and Peirce did consider division as part of their work on the logic of classes, although this aspect of their work has been omitted from this project). How might division be defined for the logic of classes, and what complications arise with regards to division of classes beyond those encountered with division in standard arithmetic?

5   The Algebra of Logic, Set Theory and Boolean Algebra Today

And what has become of Boole's Algebra of Logic today? In one guise, the symbolic representation of classes (sets) initiated by Boole is alive and well in the notation and properties of elementary set theory. *Notationally,* today's set theory looks quite different from that of Boole, Venn or Peirce, as illustrated in the table on the following page. As you examine that table, take note of the following additions or refinements which have also taken place within the system over the intervening years:

1. A clear distinction between individual *elements* (represented by lower case letters) and *sets* of elements (represented by upper case letters), where '$x \in A$' indicates that the element x is an element (or member) of the set A.

2. Explicit recognition of the relation 'set inclusion' between sets, where '$A \subseteq B$' indicates that the set A is a subset (or part) of the set B (i.e., for every element x, if $x \in A$, then $x \in B$).

3. The removal of the previous restriction pertaining to $A - B$ (relative difference), so that B no longer is required to be a subset (or part) of A.

4. Explicit recognition of the 'exclusive or' as a distinct operation (symmetric difference), where $A \triangle B$ is the set consisting of all elements that are in A or in B, but not both.

The removal of the restriction pertaining to the (relative) difference $A - B$ is a particular departure from the thinking of Boole, Venn and Peirce, one which may seem to lead to the puzzling situation in which elements are removed from A which were never there in the first place. For example, if $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$, we then have $A - B = A \cap B' = \{1, 2, 3, 4, 5\} \cap \{1, 3, 5, 7\} = \{1, 3, 5\}$. Notice, however, that today's definition of $A - B$ as the *intersection* of A with the complement of B focuses the attention on *properties* which the elements possess (i.e., the set of x's which are A's, but not B's), rather than the notion of removal

of elements. Or, to put this another way, set difference is no longer viewed as an inverse operation in any sense, as one might expect in light of the difficulties associated with this idea which this project examined.

Contemporary Set Theory and Venn's Logic of Classes

Set Concept	Current Notation & Definition	Venn's Notation	Corresponding Logical Connective
Membership Relation	$x \in A$ 'x is an element of A'	$---$	$---$
Subset Relation	$A \subseteq B$ 'A is a subset of B'	$---$	$---$
Universal Set	U	1	$---$
Empty Set	$\emptyset = \{\,\}$	0	$---$
Union	$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$	$A + B$	Inclusive 'or'
Intersection	$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$	AB	'and'
Complement	$A' = \{x \in U \mid x \notin A\}$	$1 - A = A'$	'not'
Relative Difference	$A - B = A \cap B'$ $\quad = \{x \in U \mid x \in A \text{ and } x \notin B\}$	$A - B$	$---$
Symmetric Difference	$A \triangle B = (A - B) \cup (B - A)$	$---$	Exclusive 'or'

With the exception of the refinements listed above and some (superficial) differences in notation, however, today's elementary set theory is merely a differently dressed version of the logic of classes as it was presented by Venn. In particular, Venn's interpretation of $+$ as an inclusive 'or' with no double counting of elements and without a restriction of its applicability to disjoint sets is identical to today's interpretation of set union. In applications of set theory where the double counting of elements could present a difficulty — such as the study of probability where the question 'how many elements in A?' is of importance — mathematicians now rely on the distinction between the

collection of elements as a single *set A*, and the cardinality (or size) of that collection as a *number*. Denoting the cardinality of set A by $|A|$, for example, it is easy to see that the (numerical) formula $|A \cup B| = |A| + |B| - |A \cap B|$ avoids doubly counting the elements that belong to both A and B. On the other hand, the (alleged) equality $A \cup B = A \cup B - (A \cap B)$ *fails* to hold as a property of *sets*, as can be easily shown with a (counter)example.

One important aspect of Boole's original work which is (unfortunately) lost is today's set theoretic notation is the strong (symbolic) resemblance between 'standard algebra' and the 'algebra of logic' which Boole was able to exploit to such good effect in the development of his system. If we were to follow the direction which this algebraic aspect of Boole's system took historically once Peirce's more formal approach to logic took hold among mathematicians, we would soon arrive at a mathematical concept which is both widely studied and highly important today: *boolean algebra*. To see the relation between this abstract structure and Boole's laws of logic, consider the following definition as a representative of those found in today's undergraduate textbooks:

> *Definition* A *boolean algebra* is a six-tuple $(S, +, \cdot, ', 0, 1)$ which satisfies the following eight axioms, where S is a set, $+, \cdot$ are binary operations on S, $'$ is a unary operation on S and $0, 1$ are elements of S,
>
> 1. Commutativity of Addition: $\quad\quad\quad\quad\quad (\forall x, y \in S)(x + y = y + x)$
> 2. Commutativity of Muliplication: $\quad\quad\quad (\forall x, y \in S)(x \cdot y = y \cdot x)$
> 3. Associativity of Addition: $\quad\quad\quad\quad\quad (\forall x, y, z \in S)((x + y) + z = x + (y + z))$
> 4. Associativity of Multiplication: $\quad\quad\quad (\forall x, y, z \in S)((x \cdot y) \cdot z = x \cdot (y \cdot z))$
> 5. Distributivity of Multiplication over Addition: $\quad (\forall x, y, z \in S)(x \cdot (y + z) = x \cdot y + x \cdot z)$
> 6. Distributivity of Addition over Multiplication: $\quad (\forall x, y, z \in S)(x + y \cdot z = (x + y) \cdot (x + z))$
> 7. Identity Laws: $\quad\quad\quad\quad\quad\quad\quad\quad\quad (\forall x \in S)(x + 0 = x \,\text{and}\, x \cdot 1 = x)$
> 8. Complement Laws: $\quad\quad\quad\quad\quad\quad\quad (\forall x \in S)(x + x' = 1 \,\text{and}\, x \cdot x' = 0)$

Although the boolean algebra structure defined above is intended to be completely *abstract* in the sense that no particular meaning is ascribed to the symbols $+$, $\cdot$, etc, there are many specific *concrete* examples of boolean algebras of interest from both a theoretical point of view and from a practical point of view, especially in computer science and engineering. Elementary set theory is itself one such concrete example. To see this, let S be the set of all subsets[10] of a given universal set U. The eight boolean algebra axioms are then translatable into the language of set theory as follows:

1. Commutativity of Union: $\quad\quad\quad\quad\quad\quad\quad (\forall A, B \in S)(A \cup B = B \cup A)$
2. Commutativity of Intersection: $\quad\quad\quad\quad\quad (\forall A, B \in S)(A \cap B = B \cap A)$
3. Associativity of Union: $\quad\quad\quad\quad\quad\quad\quad (\forall A, B, C \in S)((A \cup B) \cup C = A \cup (B \cup C))$
4. Associativity of Intersection: $\quad\quad\quad\quad\quad (\forall A, B, C \in S)((A \cap B) \cap C = A \cap (B \cap C))$
5. Distributivity of Intersection over Union: $\quad (\forall A, B, C \in S)(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. Distributivity of Union over Intersection: $\quad (\forall A, B, C \in S)(A \cup (B \cap C) = (A \cup B) \cap (A \cup C))$
7. Identity Laws: $\quad\quad\quad\quad\quad\quad\quad\quad\quad (\forall A \in S)(A \cup \emptyset = A \,\text{and}\, A \cap U = A)$
8. Complement Laws: $\quad\quad\quad\quad\quad\quad\quad (\forall A \in S)(A \cup A' = U \,\text{and}\, A \cap A' = \emptyset)$

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[10]The set S of all subsets of U is also called the 'power set of U' and denoted $\mathcal{P}(U)$. For example, if $U = \{a, b, c\}$, then $\mathcal{P}(U) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ In general, $|\mathcal{P}(U)| = 2^{|U|}$, as illustrated in this example.

Because the operations of set theory do, in fact, satisfy these eight properties, we can conclude that elementary set theory is a particular example of a boolean algebra. Accordingly, it is now possible to translate any theorem from the general theory of an abstract boolean algebra into a specific property for sets; the following table gives this translation for just a few of these theorems.

Theorem/Property	Boolean Algebra Version For all $x, y \in S$:	Set Theoretic Version For all sets $A, B \subseteq U$:
DeMorgan's Laws	$(x + y)' = x'y'$	$(A \cup B)' = A' \cap B'$
	$(xy)' = x' + y'$	$(A \cap B)' = A' \cup B'$
Absorption Laws	$x + xy = x$	$A \cup (A \cap B) = A$
	$x(x + y) = x$	$A \cap (A \cup B) = A$
Idempotent Laws	$x + x = x \cdot x = x$	$A \cup A = A \cap A = A$
Involution Law	$(x')' = x$	$(A')' = A$
Bound Laws	$x + 1 = 1$	$A \cup U = U$
	$x \cdot 0 = 0$	$A \cap \emptyset = \emptyset$

In a similar fashion, the fundamental of *every* concrete example of a boolean algebra can be obtained, a fact which renders the *abstract structure* of a boolean algebra not only an interesting object in its own right, but a powerful tool for theoretical and applied practitioners alike.

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6   Notes to the Instructor

This project is designed for an introductory or intermediate course in discrete or finite mathematics that includes a study of elementary set theory. Without explicitly introducing modern notation for operations on sets, the project develops a modern understanding of these operations and their basic properties within the context of early efforts to develop a symbolic algebra for logic. Although the project focuses on what would now be called 'introductory set theory,' it also lays the ground work for a more abstract discussion of boolean algebra as a discrete axiomatized structure. Accordingly, this project may also be used as an introduction to one or both of the companion projects described below in any course which considers boolean algebra from either a mathematical or computer science perspective.

Beginning with Boole's writings on the use of symbolic algebra to represent logical classes (Section 2), this project introduces the operations of logical addition (i.e., set union), logical multiplication (i.e., set intersection) and logical difference (i.e., set difference) and examines certain restrictions placed on their use by Boole which have since been lifted. The basic laws governing these operations are also introduced, as they were developed and justified by Boole; these justifications relied in part on his definitions of the operations and in part on the analogy of his symbols with those 'standard algebra.' The project then follows refinements to Boole's system made by Venn (Section 3) and Peirce (Section 4), with the level of abstraction steadily increasing through these sections. The project concludes (Section 5) with a summary of how Boole's 'Algebra of Logic' relates to elementary set theory as it is is typically presented today, and discusses how elementary set theory (when viewed as an algebraic structure) serves as a concrete example of a boolean algebra. Standard (undergraduate) notation and properties for set theory operations in use today are included and compared to standard (undergraduate) notation and axioms for a boolean algebra in that section. Issues related to language use and set operations which pose difficulties for many students, but which are ignored or unrecognized by contemporary textbook authors, are also explicitly considered in the writings of Boole and Venn (Sections 3 and 4), and further explored through the project question in those sections.

By following the refinements made to Boole's symbolic algebra in the hands of Venn and Peirce, this project provides students with an opportunity to witness first-hand *how* the process of developing and refining a mathematical system plays out, the ways in which mathematicians make and explain their choices along the way, and how standards of rigor in these regards have changed over time. Thus, in addition to developing the properties of set theory as a specific concrete example of a boolean algebra, the project is able to explore a variety of mathematical themes, including the notion of an inverse operation, the concept of duality, issues related to mathematical notation, and standards of rigor and proof. By following one or more of these themes through the project, instructors have considerable leeway in tailoring the project to their goals for a particular group of students.

Two companion projects on boolean algebra are also available as follow-up to this introductory project, either or both of which could also be used independently of the Boole-Venn-Peirce project. Additionally, either of the two companion projects could be used as a preliminary to or as a follow-up to the other companion project.

The companion project "Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization" explores the early axiomatization of boolean algebra as an abstract structure through readings from Huntington's 1904 paper *Sets of Independent Postulates for the Algebra of Logic*. In addition to introducing the now standard axioms for the boolean algebra structure, the project illustrates how to use these postulates to prove some basic properties of boolean algebras. Specific project questions also provide students with practice in using symbolic notation, and

encourage them to analyze the logical structure of quantified statements. The project also examines Huntington's use of the two-valued Boolean algebra on $K = \{0, 1\}$ — first studied by George Boole in his work on the logic of classes — as a model to establish the *independence* and *consistency* of one of his postulate sets. The final section of the project discusses modern (undergradute) notation and axioms for boolean algebras, and provides several practice exercises to reinforce the ideas developed in the earlier sections.

The companion project "Applying Boolean Algebra to Circuit Design: Claude Shannon," based on Shannon's ground-breaking paper *A Symbolic Analysis of Relay and Switching Circuits*, begins with a concise overview of the two major historical antecedents to Shannon's work: Boole's original work in logic and Huntington's work on axiomatization. The project then develops standard properties of a boolean algebra within the concrete context of circuits, and provides students with practice in using these properties to simplify boolean expressions. The two-valued boolean algebra on $K = \{0, 1\}$ again plays a central role in this work. The project closes with an exploration the concept of a 'disjunctive normal form' for boolean expressions, again within the context of circuits.

Implementation with students of any of these projects may be accomplished through individually assigned work, small group work and/or whole class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project.